Controls 101

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October 11, 2015

Agenda

What is Control Theory?

- Given some system

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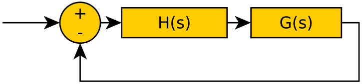
- Can we use feedback to control?



- Make an unstable system stable
- Make a stable system behave more like we want
- Can we make it robust to unmodeled plant behavior?
- Can we *guarantee* good behavior?

Complex Exponentials

- Need to analyze system behavior in usual time domain (eg, how system behaves as a function of time) as well as frequency domain (eg, how system behaves as a function of frequency)
- Euler's formula: $e^{jx} = cos(x) + jsin(x)$
  - $Acos(\omega t + \phi) = T * cos(\omega t)$
  - $Ae^{j(\omega t + \phi)} = T * e^{j\omega t}$
  - $T = Ae^{j\phi}$
- T describes a phase change and amplitude change of a sinusoid at one particular frequency
- For all linear time invariant (LTI) systems input and output frequency are equal. Only phase and gain may change...

Complex Exponentials

- Given a complex exponential, $Ae^{j\phi}$, how do we solve for ϕ and A?
- Example: $4e^{j2} = 4cos(2) + 4jsin(2) = -1.66 + 3.64j$
- Amplitude is $\sqrt{Re^2 + Im^2} = \sqrt{-1.66^2 + 3.64^2} = 4$
- Phase is $atan2(Im, Re) = atan2(3.64, -1.66) = 2$

The Laplace Transform

- Laplace transform allows us to convert back and forth from time (t) to complex frequency (s) domain!
- A single complex exponential, Ae^{jx}, can describe phase and gain change at one frequency
- What we really want to know is how they change as a *function* of frequency
- Honestly, the wikipedia page is a great resource but what are the highlights of the s-domain?

The Laplace Transform

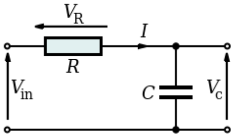
- s is shorthand for $j\omega$ where ω is the excitation frequency
- Integration in the time domain is division by s in the s-domain
- Differentiation in the time domain is multiplication by s in the s-domain
- Convolution in time domain is multiplication in the s-domain

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$

Frequency Response

Model simple RC circuit



- Time-domain:
  - $v_o = v_i - Ri$ and $i = C\dot{v}_o$
  - $\dot{v}_o = \frac{v_i - v_o}{RC}$
- S-domain:
  - $sV_o = \frac{V_i - V_o}{RC}$
  - Rearranged: $V_o = \frac{1}{sRC+1} V_i$
- $\frac{1}{sRC+1}$ is the family of complex exponentials that describe the gain/phase change from V_i to V_o
- What is gain/phase change for $RC = 1$ at frequency $\omega = 1$ [meaning $s = j\omega = j$]

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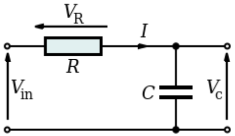

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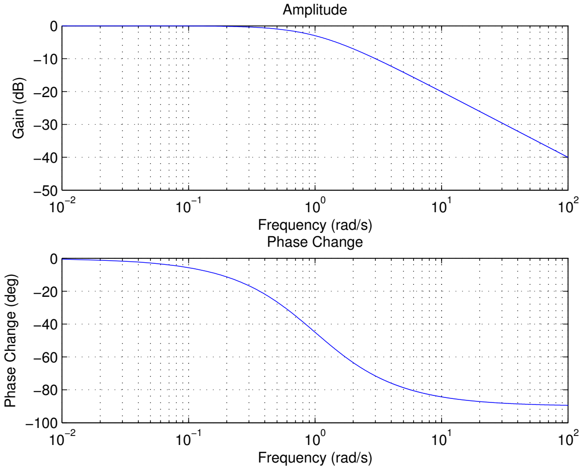
- What is gain/phase change for $RC = 1$ at frequency $\omega = 1$ [meaning $s = j\omega = j$]
  - At this frequency: $\frac{1}{sRC+1} = \frac{1}{j+1} = 0.5 - 0.5j$
  - Recall that gain is $\sqrt{0.5^2 + 0.5^2} = 0.707 = -3dB$
  - Recall that phase is $tan^{-1}(\frac{-0.5}{0.5}) = -45^\circ$
- How about the rest of the frequencies?

Frequency Response

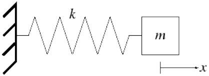


Transfer Functions

- A *transfer function* is the s-domain output divided by the input
  - $Y(s) = \frac{s+\gamma}{(s+\alpha)(s+\beta)}U(s)$
  - $\frac{Y(s)}{U(s)} = \frac{s+\gamma}{(s+\alpha)(s+\beta)}$
  - *Zeros* are values of s which make the numerator of the TF zero (eg, $-\gamma$)
  - *Poles* are values of s which make the denominator of the TF zero (eg, $-\alpha$, $-\beta$)
  - A system is stable if all its poles have a negative real component

Plant Model

- Consider a simple mass/spring system:



- Assume we have a force input that acts on the mass, u
- Goal, command the position of the block using our force input
- Time domain EOMs: $\ddot{x} = \frac{-kx+u}{m}$

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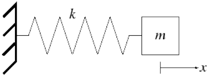

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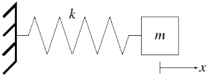

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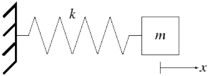

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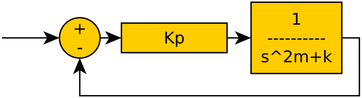
- S-domain EOMs: $s^2X = \frac{-kX+U}{m}$
- Rearranged: $\frac{X}{U} = \frac{1}{s^2m+k}$
- System is clearly not stable as poles are $s = \pm\sqrt{\frac{-k}{m}}$
  - Remember, only stable if all poles have negative real component!
- How about simple proportional control?

Proportional Control



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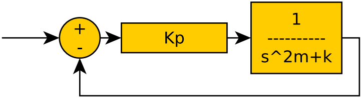
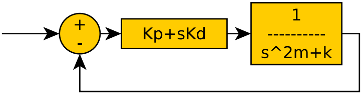


- $Y = \frac{1}{s^2m+k} K_p(U - Y)$
- $\frac{Y}{U} = \frac{K_p}{s^2m+k+K_p}$
- Roots: $s = \pm\sqrt{\frac{-k-K_p}{m}}$
- Not stable. What about PD control...

Proportional Derivative Control



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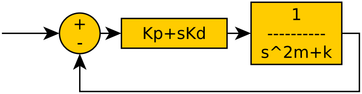
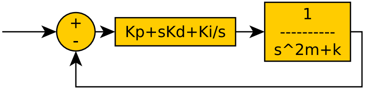


- $Y = \frac{1}{s^2m+k}(K_p + sK_d)(U - Y)$
- $\frac{Y}{U} = \frac{sK_d+K_p}{s^2m+sK_d+K_p+k}$
- Roots: $s = \frac{-K_d \pm \sqrt{K_d^2 - 4m(K_p+k)}}{2m}$
- Can force $Re(s) < 0$ through gain choices!
- What about tracking a step command, U?
  - Transfer function gain when $s = j\omega = 0$ is $\frac{K_p}{k+K_p}$. Want this to be **1**!
- What about PID control?

Proportional Integral Derivative Control



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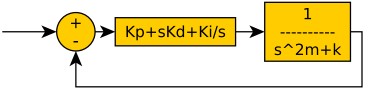
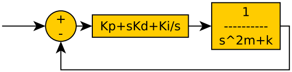


- $Y = \frac{1}{s^2m+k}(K_p + sK_d + \frac{1}{s}K_i)(U - Y)$
- $\frac{Y}{U} = \frac{s^2K_d+sK_p+K_i}{s^3m+s^2K_d+s(K_p+k)+K_i}$
- Roots are... complicated, but can be made stable!
- What about tracking a step command, U?
  - Transfer function gain when $s = j\omega = 0$ is $\frac{K_i}{K_i} = 1$!
- So how should we pick our gains?

Gain & Phase Margin

- Gain, phase margin relate to robustness to unmodeled plant behavior
- $\frac{Y}{U} = \frac{s^2 K_d + sK_p + K_i}{s^3 m + s^2 K_d + s(K_p + K_i) + k}$
- Pick $m = 1$ $k = 0.1$ $K_d = 1$ $K_p = 1$ $K_i = 1$



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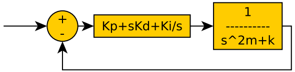
- Pick $m = 1$ $k = 0.1$ $K_d = 1$ $K_p = 1$ $K_i = 1$
- Assume there is a gain on the input of $G(s)$ of K_e
  - $\frac{Y}{U} = \frac{K_e(s^2 K_d + sK_p + K_i)}{s^3 m + s^2 K_e K_d + s(K_e K_p + k) + K_e K_i}$
  - $K_e = 0.9 = -1$ dB gives unstable poles, not awesome



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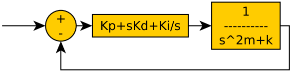
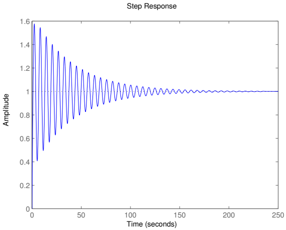
  - $K_e = 0.9 = -1$ dB gives unstable poles, not awesome
- Assume there is a phase delay on the input of $G(s)$ of $e^{j\phi}$
  - $\frac{Y}{U} = \frac{e^{j\phi}(s^2 K_d + sK_p + K_i)}{s^3 m + s^2 e^{j\phi} K_d + s(e^{j\phi} K_p + k) + e^{j\phi} K_i}$
  - $\phi \approx 5.7^\circ$ gives unstable poles,





Proportional Integral Derivative Control with LQR Gains

- Linear Quadratic Regulators (LQR) are optimal (in some sense) controllers which guarantee certain stability
  - At least 6dB of gain margin and 60° of phase margin
- Deriving, explaining LQR techniques is an L&L unto itself
- LQR suggests the following gains for our system
  - $K_d = 0.18$ $K_p = 0.017$ $K_i = 0.01$
  - Now have -301 dB of gain margin, 70 deg phase margin

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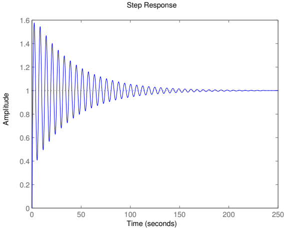
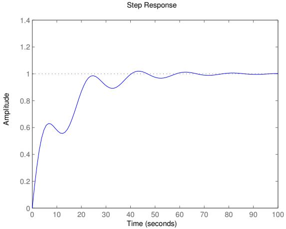


(b) LQR gains



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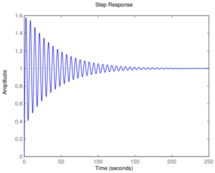

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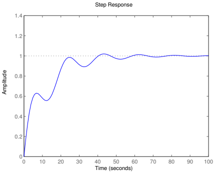

Attenuate plant input by 2 dB...

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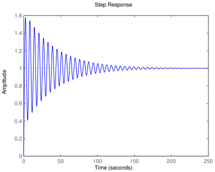

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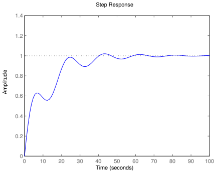

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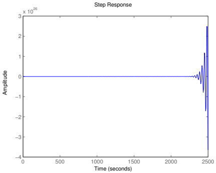

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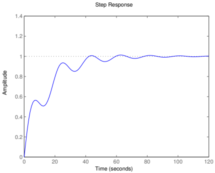

Advanced Control Techniques

- Adaptive control: Gains change to accomodate changes in plant
  - *e.g.* Control airplane where mass changes over time due to fuel burn
- Bang-Bang control: All on/off control input
  - Typically more time optimal that other approaches. "Infinite gain" in some sense.
- H-infinity control: Bound the output response for a given input disturbance
  - *e.g.* Know max wind gust, design control so that airplane pitch changes perturbs by no more than defined amount